Philosophy of Mathematics
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RUSSELL AND THE THEORY OF TYPES

The aim is to provide a general logical theory that will both:

- provide a secure foundation for mathematics; and
- solve a whole range of paradoxes.

These paradoxes include not only set-theoretic paradoxes such as Russell's, but also 'semantic' paradoxes (i.e., paradoxes involving notions like truth or definability) such as the 'Epimenides' (now usually called the Liar paradox).

The form of this Russell considers involves a man saying 'I am lying'. Actually, as Chihara points out, this probably isn't *really* paradoxical; one can just suppose that the man in question is sincerely mistaken; i.e., saying something false but which isn't a lie. So it is perhaps best to focus on the version of this in which someone says either 'I am saying something false' or 'I am saying something untrue'.

So suppose:

> Man: I am saying something untrue.

And suppose that this utterance is true; then what it says must be the case; but then it is *un*true; so it is untrue; but then what it says *is* the case; so it is true; so it is true iff it isn't.

And Russell gives a range of other paradoxes.

1. Russell's diagnosis

In each case, there is a certain 'self-reference', 'reflexiveness', or 'vicious circularity'.

Thus, Man's remark is clearly about itself (it is a quantified claim and it includes itself within its own scope). Similarly, the Russell set r is defined by means of a condition that is assumed to hold for *all* sets, including r.

And similarly, Russell argues, in the other cases too.

Another way to put the point.

- If one tries to work out whether Man's utterance is true, one is sent back to consider that very question.
- If one tries to work out whether r belongs to itself or not, one is sent back to consider that very question.

It is surely plausible that something *like* this thought must be on the right track. That is, that something like circularity is what is 'responsible' for the paradoxes.

The guiding idea behind Russell's theory: ban such vicious circularity!

2. Propositional functions

In particular, he gives a theory of 'propositional functions'. Some debate about what these are (see Chihara!) but my best guess is that they are *something like* propositions with 'holes' (or 'gaps') in them.

For example, suppose that you start with the proposition Bertrand is funny, which I think for these purposes one can (to be faithful to Russell) think of as being constructed out of the property of being funny, together with Bertrand.

Funniness(Bertrand).

If one removes Bertrand from this proposition, one gets a propositional function:

Funniness(__).

That is, one gets a 'propositional skeleton' that, given an object, yields a complete proposition. (So propositional functions are functions *in a sense*, but not strictly.)

But propositional functions are not always so simple; they can also contain logical operators like ¬, ∧, quantifiers, and so on.

3. Russell's theory

So: Russell's theory is an account of propositional functions, on which 'vicious circularity' is banned.

In particular, propositional functions are stratified into a hierarchical structure, the two fundamental ideas behind which are:

(i) a propositional function can only take as arguments things lower down in the hierarchy; and
(ii) a propositional function can only quantify over things lower down in the hierarchy.

For example, to illustrate (i), if $\varphi(x)$ is a propositional function, then it cannot take itself as argument. That is, there will be no proposition $\varphi(\varphi(x))$ (e.g., no proposition Funniness(Funniness(__)), etc.).

Similarly, to illustrate (ii), a propositional function $\varphi(x)$ of the form $\exists y\psi(x,y)$ will always be such that the quantifier $\exists y$ ranges over a lower level of the hierarchy than $\varphi(x)$ itself belongs to.

Both of these seem pretty well motivated to me (and *similarly* motivated).

Thus, (i) is motivated by the thought that the things that a propositional function applies to must be (in some sense) 'prior' to the function itself. While (ii) is motivated by the (very similar) thought that the objects in a propositional function's domain of quantification must also be 'prior' to the function itself.

To borrow Chihara's diagram (and focusing, as he does, on 1-place propositional functions) one gets the following structure:

$$\begin{array}{llllllll}
\vdots \\
T_{4.0} & T_{4.1} & T_{4.2.0} & T_{4.2.1} & T_{4.3.0} & T_{4.3.1} & T_{4.3.2.0} & T_{4.3.2.1} \\
T_{3.0} & T_{3.1} & T_{3.2.0} & T_{3.2.1} \\
T_{2.0} & T_{2.1} \\
T_{1}
\end{array}$$

T_0

I.e., each 'T_α' denotes a certain 'level' of the hierarchy; a *type* in Russell's terminology.

Thus, T_0 is the type of individuals (i.e., the initial objects one is given, which are not propositional functions).

T_1 are propositional functions of individuals which do not themselves contain quantifiers over propositional functions (either no quantifiers at all, or only over individuals). These are known as *first-order* propositional functions of individuals.

There are then two sorts of *second-order* propositional functions; these are propositional functions that contain quantifiers over first-order propositional functions, or apply to such functions. There are those that take individuals as arguments (i.e., type $T_{2.0}$) and those that take first-order propositional functions as arguments (i.e., type $T_{2.1}$).

Thus, the 'order' of a propositional function is determined by the order of the propositional functions it quantifies over (with the qualification that the order of a propositional function is always at least one greater than that of the things that it takes as arguments).

There are then three types of order 3, and so on.

Some examples:

- We belong to T_0 (of course).
- Funniness(x) is T_1.
- $\exists\varphi_1(\varphi_1(\text{Napoleon}) \wedge \varphi_1(x_0))$ is $T_{2.0}$.
- $\varphi_1(\text{Napoleon})$ is $T_{2.1}$.

4. Back to the paradoxes

4.1. Russell's paradox

Russell does not introduce sets over and above his propositional functions; although he does introduce a way of defining talk of sets as a complicated way of saying something about propositional functions.

But the essence of his solution of his paradox can be illustrated by thinking about the version for propositional functions.

Thus, naively (i.e., before Russell's hierarchical theory) one might have thought that propositional functions *can* (at least sometimes) apply to themselves. (One might have thought that PropositionalFunction(x) was one such function for example!)

One will then get a version of Russell's paradox if one considers the propositional function R(x) that applies to a propositional function φ(x) iff φ(x) does not apply to itself (i.e., iff ¬φ(φ(x))).

- But one then gets a paradox if one asks: does R apply to itself or not?
- For one seems to have R(R(x)) iff ¬R(R(x)).

But one doesn't get any such paradox on Russell's stratified approach: because propositional functions can never apply to themselves; and so there is no single propositional function that applies to *any* propositional function that does not apply to itself; rather, there will be a different 'R(x)' for each different type of propositional function; but these ‘R(x)’s will always be of higher type than the functions that they apply to.

For example, there will be a propositional function $R_{2.1}(x_1)$ that applies to a function φ_1 of type T_1 iff φ_1 does not apply to itself. (In fact, $R_{2.1}(x_1)$ will thus apply to every function in T_1.)

- But one does *not* now have that $R_{2.1}(x_1)$ applies to itself iff it does not;
- Because $R_{2.1}(x_1)$ only applies to functions of type T_1 iff they do not apply to themselves;
- And $R_{2.1}(x_1)$ is of type $T_{2.1}$.

So the paradox is blocked on this theory.

4.2. The Liar paradox

Think again about:

Man: I am saying something untrue.

Russell proposes that we analyze this as:

Man: $\exists p(S(p) \wedge \neg True(p))$,

where S(x) is a propositional function applying precisely to those propositions that Man asserts (at the relevant time).

But now let q be the proposition that Man asserts. On Russell’s theory, q will not itself be within the range of the quantifier $\exists p$. But in that case (since q is the only proposition that Man asserts at the relevant time) q will be straightforwardly (i.e., unparadoxically) false (because there is no p within the range of $\exists p$ that S(x) applies to).

Again: the paradox seems to be blocked.

5. Arithmetic in Russell’s theory

One way to do this is to mimic Frege’s approach; i.e., Frege’s construction of numbers.

Thus, numbers would be propositional functions of type $T_{2.1}$.

- 0: the propositional function applying to those propositional functions of type T_1 that apply to no individuals.
- 1: the propositional function applying to those propositional functions of type T_1 that apply to a unique individual.
- Etc.

6. Problems with the theory

(a) Complexity.

E.g., as compared with ZF set theory. But how fair is the comparison?

(b) Generalizations about propositional functions.

One cannot express claims such as: no propositional function applies to itself. So:

- hard to see how one can give general accounts of language, thought, etc.;

- hard to see how the theory itself can be stated.

Russell's solution: schematic generality. I.e., 'schematic'/'ambiguous' assertions about every level of the hierarchy.

- E.g., $p \wedge q \rightarrow p$; where 'p' and 'q' are schematic letters standing for propositions at any level of the hierarchy.

But isn't this just to add a new highest level of the hierarchy; and doesn't the problem then just emerge again?

(c) Issues with logical operators like negation.

Why are propositions of the form $\neg\neg p$ allowed? (Similarly $\exists x \forall y \varphi(x,y)$ and $\forall y \exists x \varphi(x,y)$.)

7. Sidenote: Yablo's paradox

Some paradoxes do not involve circularity:

(S0) For all $n > 0$, (Sn) is not true.
(S1) For all $n > 1$, (Sn) is not true.
⋮ ⋮
(Sm) For all $n > m$, (Sn) is not true.
⋮ ⋮

However, Russell's theory does it turns out also block paradoxes of this form (although not in virtue of the fact that it bans circularity, rather in virtue of the fact that it is 'well-founded').