Philosophy of Mathematics
PHIL 437/637, Week 11: 14 April 2015

INDETERMINACY

1. Theorems

Compactness Theorem. Let L be a first-order language, and let Φ be a set of formulas of L. Φ is satisfiable iff every finite subset of Φ is satisfiable.

Proof (from the completeness theorem). Left-to-right direction trivial. If every finite subset of Φ is satisfiable, then (by completeness) every finite subset of Φ is consistent (i.e., there is no proof of a contradiction from any finite subset of Φ). It follows that Φ is consistent. ■

Let L be a first-order language. If $\mathbf{M}$ and $\mathbf{N}$ are L-structures, then $\mathbf{M}$ is a *substructure* of $\mathbf{N}$ if the domain of $\mathbf{M}$ is a subset of the domain of $\mathbf{N}$, and the structures agree on the domain of $\mathbf{M}$. More precisely:

(i) for any n-place function letter, f, and $d_1, \ldots, d_n$ in the domain of $\mathbf{M}$, $f^{\mathbf{M}}(d_1,\ldots,d_n) = f^{\mathbf{N}}(d_1,\ldots,d_n)$;

(ii) for any n-place relation letter, R, and $d_1, \ldots, d_n$ in the domain of $\mathbf{M}$, $\langle d_1,\ldots,d_n\rangle \in R^{\mathbf{M}}$ iff $\langle d_1,\ldots,d_n\rangle \in R^{\mathbf{N}}$.

$\mathbf{M}$ is an *elementary* substructure of $\mathbf{N}$ if for any formula $\varphi(x_1,\ldots,x_n)$ and $d_1, \ldots, d_n$ in the domain of $\mathbf{M}$:

$$\mathbf{M} \models \varphi[d_1,\ldots,d_n] \text{ iff } \mathbf{N} \models \varphi[d_1,\ldots,d_n].$$

(Obviously, if $\mathbf{M}$ is an elementary substructure of $\mathbf{N}$ then they satisfy the same sentences: let φ be a sentence.)

Upward Löwenheim-Skolem Theorem. Let $\mathbf{M}$ be an infinite L-structure and κ a cardinal with $\kappa \geq |\mathbf{M}| + |L|$. There is an L-structure, $\mathbf{N}$, of which $\mathbf{M}$ is an elementary substructure, with $|\mathbf{N}| = \kappa$.

Downward Löwenheim-Skolem Theorem. Let $\mathbf{M}$ be an L-structure and let X be a subset of the domain of $\mathbf{M}$. Then there is an elementary substructure $\mathbf{N}$ of $\mathbf{M}$ with X a subset of the domain of $\mathbf{N}$ and $|\mathbf{N}| \leq |X| + |L| + \aleph_0$.[1]

2. Basic Argument for Indeterminacy

The results of §1 entail that familiar mathematical theories will have a variety of 'non-standard' models.

For example, any sound theory of arithmetic Φ (e.g., PA) will have models with 'infinite' natural numbers, i.e., natural numbers with infinitely many predecessors.

[1] For proofs see, e.g., D. Marker, *Model Theory*.

This follows from the compactness theorem. E.g., consider $\Phi \cup \{x > n$: n a natural numeral$\}$. This will be satisfiable by compactness, but any structure (and assignment) satisfying it will have to have elements in the domain with infinitely many predecessors.

Similarly, any sound theory of analysis Φ will have models with infinitesimals: positive numbers smaller than 1/n for every n.

By the downward Löwenheim-Skolem theorem, ZFC will have a countable model (as long as it is consistent).

Many (Skolem, Putnam, Field) have held that these results show that at least some of our mathematical terms are radically indeterminate: they fail to rule out these non-standard structures.

The basic line of argument is this:

- All our grasp of mathematical language can amount to is an implicit acceptance of a certain theory.
- But whatever that theory is, the results of §1 show that it will have non-isomorphic models of various sorts.
- So this grasp does not determine a unique structure.
- For example, our use of the term natural number does nothing to determine 'standard' over 'non-standard' models.
- One wants to respond: but only one structure is intended!
- But in virtue of what?
- To specify our referential intentions we just use more language, and there is no more reason to think these additions are anymore determinate.

3. Second-Order Logic

One response to the threat of indeterminacy is to claim that the problem only comes if one has an impoverished conception of logic, and a correspondingly impoverished conception of what our grasp of mathematical language amounts to.

In particular—it would be claimed—one should think of this grasp not as amounting to implicit acceptance of a *first*-order theory, but a *second*-order one. I.e., a theory that contains, in addition to familiar first-order quantifiers ranging over objects, also second-order quantifiers ranging over (something like) properties and relations (or concepts or pluralities or…) applying to the objects the first-order quantifiers range over. (Issue about pluralities and monadicity.)

More precisely, let L be a second-order language (assume that L is just a first-order language plus second-order quantifiers; i.e., no new higher-order predicate letters, etc.).

Then a 'second-order' L-structure $\mathbf{M}$ is the same thing as a 'first-order' L-structure: a domain D plus assignments to the non-logical vocabulary.

That is, D determines not just the range of the first-order quantifiers (which is just D) but also the ranges of the second-order quantifiers: the n-place second-order relation variables range over the subsets of X^n (and similarly for second-order function variables if such there be).

For example, $\forall x\exists Y\forall z(Yz \leftrightarrow z = x)$ is a second-order logical truth—satisfied by every structure—because for any domain D and $d \in D$, $\{d\}$ is a subset of D (= D^1).

Crucial fact: the analogues of the results of §1 do *not* hold for second-order logic.

For example, consider arithmetic and the second-order induction axiom:

(I2) $\forall X(X0 \wedge \forall x(Xx \rightarrow Xx{+}1) \rightarrow \forall xXx)$.

Any structure will satisfy (I2) iff its domain consists solely of '0' together with everything that can be reached from '0' by 'adding 1' a finite number of times.

That is, (I2) has no models with 'infinite' natural numbers.

Thus, second-order PA (= first-order PA but with (I2) in place of the first-order induction scheme) is *categorical*: any two models are isomorphic.

Similarly, consider the following sentence of the second-order language of set theory:

(S2) $\forall x\forall X\exists y\forall z(z \in y \leftrightarrow z \in x \wedge Xz)$.

Any model of ZFC that satisfies (S2) must be uncountable: it will contain an 'infinite set' x and must thus (by (S2)) contain distinct (by extensionality) sets corresponding to every subset of the set of 'members' of x.

In fact, second-order ZFC (ZFC but with second-order replacement in place of the first-order scheme) is not categorical but is (the next best thing) *quasi-categorical*: for any two models, one will be isomorphic to an initial segment of the other.

So… this might lead one to think that the problem raised by the results of §1 and the argument of §2 is easily solved after all: we only thought we had a problem because we were fixated on an impoverished fragment of logic.

But the response the proponents of the original indeterminacy arguments will make is as follows.

- This 'solution' simply assumes that we have a determinate grasp of second-order quantification.
- But this is challengeable in just the way in which it was challenged that we have a determinate grasp of 'natural number', 'set', etc.
- After all, all our grasp of second-order quantification comes down to is acceptance of a certain theory.
- And this theory will itself have 'non-standard' models.
- For example, models in which the second-order variables range not over *every* subset of the first-order domain, but only over some (perhaps countably many) of them.
- Of course it will be claimed that these 'unfull' second-order models are not what is intended—but why is this any more reasonable than the analogous move in the case of 'number' and 'set'?

Thus appealing to second-order logic is often regarded as cheating, and ineffective.

4. Non-Mathematical Language

An alternative strategy (which Field explores) is to try to use the non-mathematical parts of our total theory to make the mathematical portion determinate.

The downward Löwenheim-Skolem theorem suggests that this will not work in the case of set theory.

- For example, let L be our total language, including set-theoretic, other mathematical, and also non-mathematical, terms.
- Let Φ be our 'total theory', and let **M** be an L-structure claimed to be the unique one compatible with our use of L.
- So presumably the domain of L is uncountable.
- Let X be the set of concrete objects (and suppose that X is countable, e.g.).
- There will then be a countable elementary substructure **N** of **M** that includes X in its domain.
- **N** will interpret all of our non-mathematical language as intended (with respect to concrete objects) but will contain only countably many 'sets'.
- Thus it seems that appealing to non-mathematical language will not help in the set-theoretic case.

But no such argument seems available to show that there can be an interpretation of Φ that gives the 'right' interpretation to the non-mathematical parts, but that has 'non-standard' numbers (with infinitely many predecessors).

To illustrate why: Φ might contain a sentence saying that there is an isomorphism between (e.g.) the planets and the miles-from-earth ordering, and the natural numbers and <. (Field gives a somewhat more plausible example, but the idea is the same!)

Then the non-mathematical parts of Φ will fix a unique structure that the 'numbers' must possess.

So the problem, at least as focused on 'natural number', 'finite', etc., has been solved, Field suggests.

Of course, one might wonder (with Putnam, essentially) *why* exactly indeterminacy arguments of the same general sort cannot be used to show that our *non*-mathematical language is determinate?

In a nutshell…

> Field: the *difference* is causation.
> Putnam: that's just *more theory*!
> Field: you're missing the point.

But, even if the general *not*-just-more-theory move is reasonable, one might suspect causation in and of itself is not enough to rule-out non-standard interpretations of our non-mathematical terms. E.g., planet-surfaces vs planets, sense impressions of Venus vs Venus. Also what about '='? Connectives?

Standard move at this sort of juncture (e.g., Lewis): the reference of 'planet', 'Venus' etc. is the most 'natural', or the 'simplest'—or the one with the most '(referential) magnetism'!

But if that *is* the right account of how ‘planet’, ‘=’, etc. get their referents, then cannot this be used to challenge the original argument for indeterminacy?

That is, why can we not argue that the natural number structure is most natural; as are non-standard interpretations of the second-order quantifiers?